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TRANSIENT THERMAL BEHAVIOR OF MULTILAYER MEDIA: MODELING AND APPLICATION TO STRATIFIED MOULDS

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Transient and steady-state heat transfer in multilayer media is investigated by the thermal quadrupole method. A semi-analytical solution is proposed for the cases of layers parallel or orthogonal to the main heat-flux direction. The principal application is the study of the effect of the brazing metal used in stratified steel moulds.

Introduction. Thanks to technological improvements, the use of stratified media has become more and more important. A better understanding of their behavior is needed. Although there is an enormous number of mechanical studies, only a few authors took an interest in heat-transfer modeling and most often the study was limited to the case of two layers. As far as the heat transfer is concerned, the main goal is to know if a spatially periodic heterogeneous medium could be represented in a suitable way by a homogeneous medium and in this case to determine the equivalent properties.

The first part of the study presents the modeling of the multilayer to obtain its thermal behavior in terms of temperature and heat flux. Two configurations are considered: layers parallel and orthogonal to the main heat-flux direction. Then the real case of stratified moulds used in rapid tooling is investigated. Results for the steady state, involving thermal resistance, and for the transient case are presented and discussed.

Description of the Model. The model proposed here is based on the quadrupole formulation [1, 2]. This technique has been developed and has been widely used in the last few years [3, 4], even in the case of transient coupled radiative-conductive heat transfer with anisotropic scattering [5]. This method is commonly used to solve ordinary differential equations in the Laplace domain. It provides a transfer matrix for the medium that linearly links the input temperature-heat flux column vector at the front side and the output vector at the back side. This model gives results in the Laplace domain. A numerical algorithm (Stehfest [6] or de Hoog [7]) permits one to obtain the temperature and the heat flux as functions of time.

A short presentation of this method is proposed below (for further details and explanations in the most complicated cases, see [1, 2, 5]). Let us assume a one-dimensional Cartesian coordinate system. The transient heat-transfer equation is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x, t)}{\partial t} .$$

Application of the Laplace transform gives

$$\frac{d^2 \theta (x, p)}{dx^2} = \frac{p}{\alpha} \theta (x, t) .$$

The heat flux in the image domain is

$$\phi (x, p) = - \lambda S \frac{d\theta (x, p)}{dx} .$$

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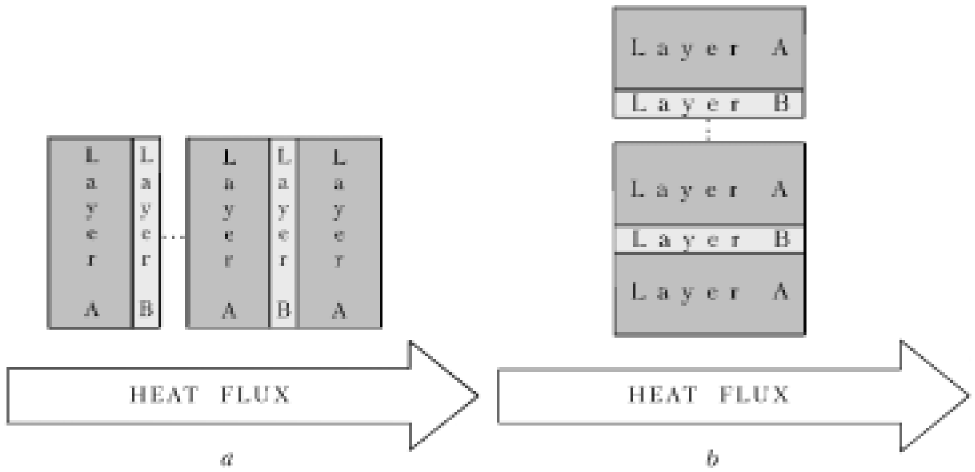


Fig. 1. Scheme of the medium in the case of orthogonal (*a*) and parallel (*b*) heat transfer.

The last two equations can be written in a matrix formulation:

$$\frac{d}{dx}\begin{pmatrix}\theta(x,p)\\ \phi(x,p)\end{pmatrix}=\begin{pmatrix}0 & -\dfrac{1}{\lambda S}\\ -\dfrac{\lambda S p}{\alpha} & 0\end{pmatrix}\begin{pmatrix}\theta(x,p)\\ \phi(x,p)\end{pmatrix}.$$

This can be summarized in the form

$$\frac{d\overline{X}}{dx}=-M\overline{X}\,.$$

The solution of this equation is

$$\overline{X}(x,p)=\exp(-Mx)\,\overline{X}(0,p)\,,$$

where

$$\exp(-Mx)=\begin{pmatrix}\cosh(\sqrt{p/\alpha}\,x) & -\sinh(\sqrt{p/\alpha}\,x)/(\lambda\sqrt{p/\alpha}\,S)\\ -\lambda\sqrt{p/\alpha}\,S\sinh(\sqrt{p/\alpha}\,x) & \cosh(\sqrt{p/\alpha}\,x)\end{pmatrix}.$$

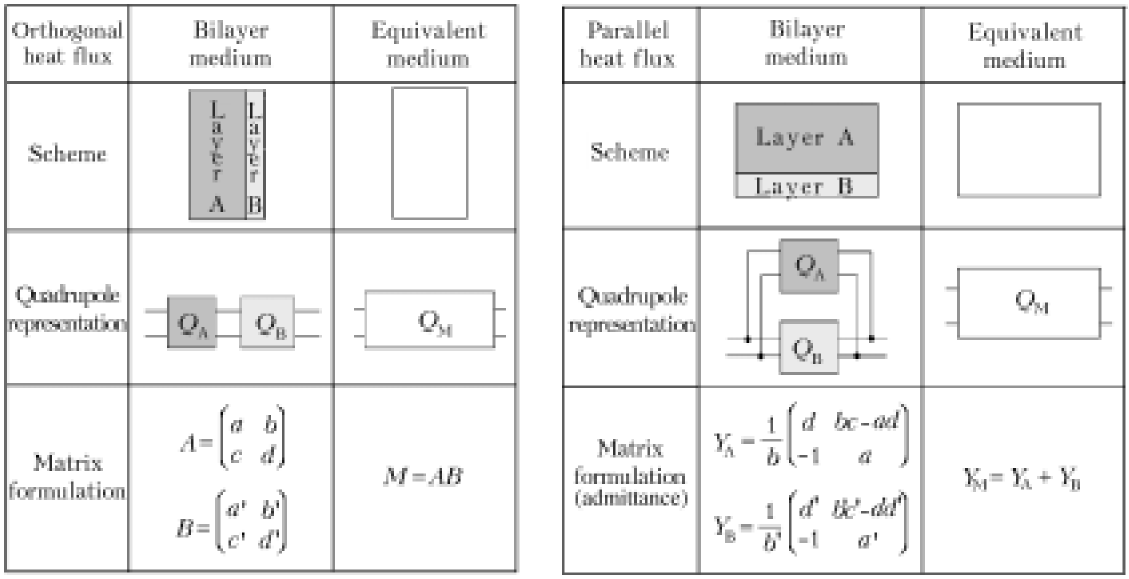
As a result of this, the quadrupole formulation can be written:

$$\begin{pmatrix}\theta(0,p)\\ \phi(0,p)\end{pmatrix}=\begin{pmatrix}\cosh(\sqrt{p/\alpha}\,x) & \sinh(\sqrt{p/\alpha}\,x)/(\lambda\sqrt{p/\alpha}\,S)\\ \lambda\sqrt{p/\alpha}\,S\sinh(\sqrt{p/\alpha}\,x) & \cosh(\sqrt{p/\alpha}\,x)\end{pmatrix}\begin{pmatrix}\theta(x,p)\\ \phi(x,p)\end{pmatrix}.$$

First configuration. In this case, the main direction of the heat flux is orthogonal to the layers, as schematically presented in Fig. 1a. Let us consider a simple *medium composed of two layers* (Fig. 2a), which are represented by matrices A and B:

$$A=\begin{pmatrix}\cosh u & \dfrac{\sinh u}{\Delta}\\ \Delta\sinh u & \cosh u\end{pmatrix}, \tag{1}$$

$$B=\begin{pmatrix}\cosh v & \dfrac{\sinh v}{\delta}\\ \delta\sinh u & \cosh v\end{pmatrix}, \tag{2}$$



a *b*

Fig. 2. The bilayer medium in the case of orthogonal (*a*) and parallel (*b*) heat transfer.

where

$$u=k_A e_A\,,\quad \Delta=\lambda_A k_A S_A\,,\quad k_A=\sqrt{p/\alpha_A}\,,$$

$$v=k_B e_B\,,\quad \delta=\lambda_B k_B S_B\,,\quad k_B=\sqrt{p/\alpha_B}\,.$$

To obtain the equivalent matrix M that models the thermal behavior of the heterogeneous medium, the product of the two matrices A and B must be obtained:

$$M=\begin{pmatrix} \cosh u\cosh v+\dfrac{\delta}{\Delta}\sinh u\sinh v & \dfrac{1}{\delta}\cosh u\sinh v+\dfrac{1}{\Delta}\sinh u\cosh v \\ \Delta\sinh u\cosh v+\delta\cosh u\sinh v & \dfrac{\Delta}{\delta}\sinh u\sinh v+\cosh u\cosh v \end{pmatrix}. \tag{3}$$

Now a *multilayer medium* is investigated. The equivalent matrix N of this assembly is given by the following formula:

$$N=M^n A\,. \tag{4}$$

In order to evaluate M^n, it is of interest to determine the eigenvalues λ_i and the eigenvectors $\mathbf{V}_i(x_i,\ y_i)$ of the matrix M. The explicit expression of the matrix N of the multilayer medium is

$$N=PD^nP^{-1}A \tag{5}$$

or

$$N=\frac{1}{x_1y_2-x_2y_1}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6}$$

where

TABLE 1. Relation between the Transfer (N) and Admittance (Y_N) Matrices

N	Y_N
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\frac{1}{b}\begin{pmatrix} d & -1 \\ -1 & a \end{pmatrix}$
$\frac{1}{m}\begin{pmatrix} -s & -1 \\ m^2-ls & -l \end{pmatrix}$	$\begin{pmatrix} l & m \\ r & s \end{pmatrix}$

$$x_i = \frac{1}{\delta}\cosh u \sinh v + \frac{1}{\Delta}\sinh u \cosh v\,,$$

$$y_i = \lambda_i - \cosh u \cosh v - \frac{\delta}{\Delta}\sinh u \sinh v\,,$$

$$\lambda_i = \cosh u \cosh v + \left(\frac{\delta}{2\Delta} + \frac{\Delta}{2\delta}\right)\sinh u \sinh v \pm$$

$$\pm\left(\sinh^2 u \sinh^2 v\left[1 + \left(\frac{\delta}{2\Delta} + \frac{\Delta}{2\delta}\right)^2\right] + \left(\frac{\delta}{\Delta} + \frac{\Delta}{\delta}\right)\cosh u \cosh v \sinh u \sinh v\right)^{1/2}.$$

Second configuration. In this case, the main direction of the heat flux is parallel to the layers, as shown in Fig. 1b.

Let us consider a simple *medium composed of two layers* (Fig. 2b). Two matrices (Y_A and Y_B respectively) that are called admittance matrices and that represent each layer must be considered:

$$Y_A = \frac{1}{(\sinh u)/\Delta}\begin{pmatrix} \cosh u & -1 \\ -1 & \cosh u \end{pmatrix}, \tag{7}$$

$$Y_B = \frac{1}{(\sinh v)/\delta}\begin{pmatrix} \cosh v & -1 \\ -1 & \cosh v \end{pmatrix}. \tag{8}$$

The relations between the transfer matrices A and B and the admittance matrices Y_A and Y_B are seen from Table 1.

To obtain the equivalent admittance matrix Y_M for modeling the thermal behavior of the heterogeneous medium, the sum of the two admittance matrices Y_A and Y_B must be found:

$$Y_M = \begin{pmatrix} \dfrac{(\cosh u \sinh v)/\delta + (\cosh v \sinh u)/\Delta}{\sinh u \sinh v/(\Delta\delta)} & -\dfrac{(\sinh v)/\delta + (\sinh u)/\Delta}{\sinh u \sinh v/(\Delta\delta)} \\ -\dfrac{(\sinh v)/\delta + (\sinh u)/\Delta}{\sinh u \sinh v/(\Delta\delta)} & \dfrac{(\cosh u \sinh v)/\delta + (\cosh v \sinh u)/\Delta}{\sinh u \sinh v/(\Delta\delta)} \end{pmatrix}, \tag{9}$$

$$Y_M = \begin{pmatrix} \Delta \coth u + \delta \coth v & \dfrac{-\Delta \sinh v - \delta \sinh u}{\sinh u \sinh v} \\ \dfrac{-\Delta \sinh v - \delta \sinh u}{\sinh u \sinh v} & \Delta \coth u + \delta \coth v \end{pmatrix}. \tag{10}$$

Then the transfer matrix M can be written as

$$M = \begin{pmatrix} \dfrac{\Delta \cosh u \sinh v + \delta \sinh u \cosh v}{\Delta \sinh v + \delta \sinh u} & \dfrac{\sinh u \sinh v}{\Delta \sinh v + \delta \sinh u} \\ \dfrac{(\Delta \cosh u \sinh v + \delta \sinh u \cosh v)^2 - (\Delta \sinh v + \delta \sinh u)^2}{\sinh u \sinh v \, (\Delta \sinh v + \delta \sinh u)} & \dfrac{\Delta \cosh u \sinh v + \delta \sinh u \cosh v}{\Delta \sinh v + \delta \sinh u} \end{pmatrix}. \quad (11)$$

Now a *multilayer medium* is investigated. The equivalent admittance matrix Y_{N} of this assembly is given by the following formulas:

$$Y_{\mathrm{N}} = (n+1)\, Y_{\mathrm{A}} + nY_{\mathrm{B}} \,, \quad (12)$$

$$Y_{\mathrm{N}} = \begin{pmatrix} (n+1)\, \Delta \coth u + n\delta \coth v & \dfrac{-(n+1)\, \Delta \sinh v - n\delta \sinh u}{\sinh u \sinh v} \\ \dfrac{-(n+1)\, \Delta \sinh v - n\delta \sinh u}{\sinh u \sinh v} & (n+1)\, \Delta \coth u + n\delta \coth v \end{pmatrix}. \quad (13)$$

Then the expression for the transfer matrix N (obtained from the admittance matrix Y_N as explained previously) is

$$N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}, \quad (14)$$

where

$$n_{11} = \frac{(n+1)\, \Delta \cosh u \sinh v + n\delta \sinh u \cosh v}{(n+1)\, \Delta \sinh v + n\delta \sinh u} \,,$$

$$n_{12} = \frac{\sinh u \sinh v}{(n+1)\, \Delta \sinh v + n\delta \sinh u} \,,$$

$$n_{21} = \frac{\left\{(n+1)\, \Delta \cosh u \sinh v + n\delta \sinh u \cosh v\right\}^2 - \left\{(n+1)\, \Delta \sinh v + n\delta \sinh u\right\}^2}{\sinh u \sinh v \left\{(n+1)\, \Delta \sinh v + n\delta \sinh u\right\}} \,,$$

$$n_{22} = n_{11} \,.$$

Application to a Real Case: Stratified Moulds. In rapid prototyping, steel layers are brazed together to obtain stratified moulds. In this case, the layer A is steel and the layer B is the brazed metal. It is interesting to know the thermal behavior of the mould for several reasons. For instance, the quality of the piece made with this mould or the lifetime of the mould highly depends on the temperature and the heat flux in the mould. The reference case is the mould made of steel only and the purpose is to know whether the spatially periodic heterogeneous medium, i.e., the brazed layers, could be represented in a suitable way by a homogeneous medium and if this is the case to determine the equivalent properties. The geometrical and thermophysical properties of the multilayer medium used for the simulations are given in Tables 2 and 3.

Results: Temperature and heat flux within the multilayer medium. The s*teady-state case* corresponds to the limit of the previous expressions when p tends to zero. To validate the model based on the quadrupole formulation presented earlier, the terms of matrix M (Eqs. (3) and (11)) must be considered when p tends to zero.

The elements of matrix in Eq. (3) are presented as

$$m_{11} = \cosh u \cosh v + \frac{\delta}{\Delta} \sinh u \sinh v \underset{p \to 0}{\sim} 1 \,,$$

TABLE 2. Values of the Geometrical Parameters

Material	e, m		S, m^2	
	Case 1	Case 2	Case 1	Case 2
Steel	$6{\cdot}10^{-3}$	$4{\cdot}10^{-1}$	$2.4{\cdot}10^{-1}$	$2.4{\cdot}10^{-3}$
Brazing metal	$4{\cdot}10^{-7}$	$4{\cdot}10^{-1}$	$2.4{\cdot}10^{-1}$	$1.6{\cdot}10^{-7}$

TABLE 3. Values of the Thermophysical Parameters

Material	λ, W·m^{-1}·K^{-1}	α, m^2·sec^{-1}	$1/\sqrt{\alpha}$, m^{-1}·sec$^{1/2}$
Steel	15	$3.98{\cdot}10^{-6}$	501.198
Brazing metal	370	$1.0614{\cdot}10^{-4}$	97.065

$$m_{22} = \frac{\Delta}{\delta} \sinh u \sinh v + \cosh u \cosh v \underset{p\to 0}{\sim} 1 ,$$

$$m_{12} = \frac{1}{\delta} \cosh u \sinh v + \frac{1}{\Delta} \sinh u \cosh v \underset{p\to 0}{\sim} v/\delta + u/\Delta \underset{p\to 0}{\sim} \frac{e_A}{\lambda_A S_A} + \frac{e_B}{\lambda_B S_B} ,$$

$$m_{21} = \Delta \sinh u \cosh v + \delta \cosh u \sinh v \sim \Delta u + \delta v \underset{p\to 0}{\sim} (\rho_A c_A S_A e_A + \rho_B c_B S_B e_B)\, p \underset{p\to 0}{\sim} 0$$

and for Eq. (11) as

$$m_{11} = \frac{\Delta \cosh u \sinh v + \delta \sinh u \cosh v}{\Delta \sinh v + \delta \sinh u} \underset{p\to 0}{\sim} 1 ,$$

$$m_{22} = \frac{\Delta \cosh u \sinh v + \delta \sinh u \cosh v}{\Delta \sinh v + \delta \sinh u} \underset{p\to 0}{\sim} 1 ,$$

$$m_{12} = \frac{\sinh u \sinh v}{\Delta \sinh v + \delta \sinh u} \underset{p\to 0}{\sim} \frac{e_A e_B}{\lambda_A S_A e_B + \lambda_B S_B e_A} \underset{p\to 0}{\sim} \frac{1}{\lambda_A S_A / e_A + \lambda_B S_B / e_B} ,$$

$$m_{21} = \frac{(\Delta \cosh u \sinh v + \delta \sinh u \cosh v)^2 - (\Delta \sinh v + \delta \sinh u)^2}{\sinh u \sinh v\, (\Delta \sinh v + \delta \sinh u)} \underset{p\to 0}{\sim} 0 .$$

Then $\theta_S = \theta_E + m_{12}\varphi_E$ for p approaching zero takes the following form:

$$\theta_S = \theta_E + \left\{ \frac{e_A}{\lambda_A S_A} + \frac{e_B}{\lambda_B S_B} \right\} \phi_E \quad \text{in case } 1 ,$$

$$\theta_S = \theta_E + \left\{ 1/(\lambda_A S_A / e_A + \lambda_B S_B / e_B) \right\} \phi_E \quad \text{in case } 2 ,$$

where

$$\theta_E = \theta\,(x = 0) , \quad \theta_S = \theta\,(x = e = e_A + e_B) , \quad \phi_E = \phi\,(x = 0) , \quad \phi_S = \phi\,(x = e = e_A + e_B) .$$

These expressions represent the well-known relation for the steady-state case $\Delta\theta = R\varphi$, where R is the equivalent resistance of the medium ($R = R_A + R_B$ in case 1 and $R = 1/(1/R_A + 1/R_B)$ in case 2). For a multilayer, the expressions become

TABLE 4. Equivalent Thermal Resistance for Multilayer and Steel Medium without Brazing

Resistance characteristic	*n*					
	10		20		Bilayer	
	Orthogonal	Parallel	Orthogonal	Parallel	Orthogonal	Parallel
R, K·W^{-1}	$1.83334 \cdot 10^{-2}$	1.00859	$3.50001 \cdot 10^{-2}$	$5.28273 \cdot 10^{-1}$	$1.66667 \cdot 10^{-3}$	$1.10929 \cdot 10$
R_s, K·W^{-1}	$1.83344 \cdot 10^{-2}$	1.01004	$3.50022 \cdot 10^{-2}$	$5.29067 \cdot 10^{-1}$	$1.66678 \cdot 10^{-3}$	$1.11104 \cdot 10$
ΔR, K·W^{-1}	$1.06607 \cdot 10^{-6}$	$1.44658 \cdot 10^{-3}$	$2.13213 \cdot 10^{-6}$	$7.93757 \cdot 10^{-4}$	$1.06607 \cdot 10^{-7}$	$1.75009 \cdot 10^{-2}$
$\Delta R / R_s$	$5.81455 \cdot 10^{-5}$	$1.43220 \cdot 10^{-3}$	$6.09142 \cdot 10^{-5}$	$1.50030 \cdot 10^{-3}$	$6.39597 \cdot 10^{-5}$	$1.57519 \cdot 10^{-3}$

TABLE 5. Ratios of Thermal Resistances of Multilayer and Bilayer

Resistance characteristic	*n*			
	10		20	
	Orthogonal	Parallel	Orthogonal	Parallel
R/R_{bi}	$1.0999997297 \cdot 10$	$1.099835826 \cdot 10$	$2.0999997297 \cdot 10$	$2.0998358255 \cdot 10$
R_s/R_{bi}	$1.0999933338 \cdot 10$	$1.0999933338 \cdot 10$	$2.0999933338 \cdot 10$	$2.09999 \cdot 10$
Difference	$-6.3959527120 \cdot 10^{-5}$	$1.5750824640 \cdot 10^{-3}$	$-6.3959527122 \cdot 10^{-5}$	$1.57508 \cdot 10^{-3}$
Relative difference	$-5.8145377027 \cdot 10^{-6}$	$1.4319018267 \cdot 10^{-4}$	$-3.0457014359 \cdot 10^{-6}$	$7.50042 \cdot 10^{-5}$

$$\theta_S = \theta_E + \left\{ (n+1) \frac{e_A}{\lambda_A S_A} + n \frac{e_B}{\lambda_B S_B} \right\} \phi_E \quad \text{in case 1 ,}$$

$$\theta_S = \theta_E + \left\{ 1/((n+1)\, \lambda_A S_A / e_A + n \lambda_B S_B / e_B) \right\} \phi_E \quad \text{in case 2 .}$$

Numerical results for the thermal resistance are presented in Table 4. The media are composed of ten or twenty brazed layers (eleven or twenty-one steel layers). The reference case of a bilayer is also given in the table. It must be emphasized that the absolute differences $\Delta R = R_s - R$ between the equivalent thermal resistance of the multilayer (steel and brazing) R and the thermal resistance of a medium without brazing (steel only) R_s are very small (about 10^{-6} K·W^{-1} in case 1 (orthogonal heat transfer) and 10^{-3} K·W^{-1} in case 2 (parallel heat transfer)). This is essentially due to the fact that the thermal resistance of the steel layer ($1.6 \cdot 10^{-3}$ K·W^{-1}) is very high as compared to the resistance of the brazing ($4.5 \cdot 10^{-9}$ K·W^{-1}). The relative differences are about 10^{-5} and 10^{-3} in cases 1 and 2, respectively.

As a consequence, for a given difference of temperatures, the heat flux through the media composed of the multilayer (steel and brazing) is quite the same as in the case without brazing layers (this is more true when the heat transfer is orthogonal to the layers).

A further detail appears when the ratio of the thermal resistances of the multilayer and bilayer is considered: the value of this ratio is very close to the number of layers involved in the medium (Table 5). As a consequence, knowledge of the thermal resistance of a bilayer and the number of layers of the multilayer allows one to get easily and quickly a good and realistic idea of what will happen in the case of the multilayer.

Then the *transient case* is investigated. Let us consider the first configuration where the main direction of the heat flux is orthogonal to the layers. It is of importance to note our application in order to simplify the expressions of the matrices. Indeed, the layer B is very thin compared to the layer A and, as a consequence, expression (2) for the transfer matrix can be simplified in the following manner:

$$B = \begin{pmatrix} \cosh v & (\sinh v)/\delta \\ \delta \sinh v & \cosh v \end{pmatrix} \approx \begin{pmatrix} 1 & e_B/(\lambda_B S_B) \\ (\rho_B c_B e_B S_B)\, p & 1 \end{pmatrix} = \begin{pmatrix} 1 & R_B \\ C_B p & 1 \end{pmatrix}.$$

In the case of a bilayer medium, expression (3) for the transfer matrix becomes

$$M = \begin{pmatrix} \cosh u + (\rho_B c_B e_B S_B)\, p\, (\sinh u)/\Delta & (e_B/(\lambda_B S_B)) \cosh u + (\sinh u)/\Delta \\ \Delta \sinh u + (\rho_B c_B e_B S_B)\, p \cosh u & (e_B/(\lambda_B S_B))\, \Delta \sinh u + \cosh u \end{pmatrix},$$

i.e.,

$$M = \begin{pmatrix} 1 + \dfrac{\rho_B c_B S_B}{\rho_A c_A S_A} \dfrac{e_B e_A}{\alpha_A} p & e_B/(\lambda_B S_B) + e_A/(\lambda_A S_A) \\ (\rho_B c_B e_B S_B + \rho_A c_A e_A S_A)\, p & \dfrac{\rho_A c_A S_A}{\rho_B c_B S_B} \dfrac{e_B e_A}{\alpha_B} p + 1 \end{pmatrix} \approx \begin{pmatrix} 1 & R_B + R_A \\ (C_B + C_A)\, p & 1 \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = 1 + \sqrt{RCp}$ and $\lambda_2 = 1 - \sqrt{RCp}$ with $R = R_A + R_B$ and $C = C_A + C_B$. Two eigenvectors are $(\sqrt{RCp};\ Cp)$, $(\sqrt{RCp};\ —Cp)$.

It is now easy to evaluate the general formula of the matrix M^n for transient modeling of the multilayer:

$$M^n = \frac{1}{2} \begin{pmatrix} 1/(Cp) & 1/(Cp) \\ 1/\sqrt{RCp} & -1/\sqrt{RCp} \end{pmatrix} \begin{pmatrix} (1 + \sqrt{RCp})^n & 0 \\ 0 & (1 - \sqrt{RCp})^n \end{pmatrix} \begin{pmatrix} Cp & \sqrt{RCp} \\ Cp & -\sqrt{RCp} \end{pmatrix},$$

$$M^n = \frac{1}{2} \begin{pmatrix} (1 + \sqrt{RCp})^n + (1 - \sqrt{RCp})^n & \sqrt{R/(Cp)} \left((1 + \sqrt{RCp})^n - (1 - \sqrt{RCp})^n \right) \\ \sqrt{Cp/R} \left((1 + \sqrt{RCp})^n - (1 - \sqrt{RCp})^n \right) & (1 + \sqrt{RCp})^n + (1 - \sqrt{RCp})^n \end{pmatrix},$$

$$M^n = \begin{pmatrix} \displaystyle\sum_{k \text{ even}}^{n} C_n^k (RCp)^{k/2} & \displaystyle R \sum_{k \text{ odd}}^{n} C_n^k (RCp)^{(k-1)/2} \\ \displaystyle Cp \sum_{k \text{ odd}}^{n} C_n^k (RCp)^{(k-1)/2} & \displaystyle\sum_{k \text{ even}}^{n} C_n^k (RCp)^{k/2} \end{pmatrix}.$$

The explicit expressions of the transfer matrix

$$M^n = \begin{pmatrix} m_{11}^n & m_{12}^n \\ m_{21}^n & m_{22}^n \end{pmatrix} = \begin{pmatrix} P_1^n & RP_2^n \\ CpP_2^n & P_1^n \end{pmatrix}$$

for n = 10 and n = 20 are given in Table 6. Table 7 summarizes the expressions of the heat flux and the temperature as functions of the transfer-matrix elements

$$\begin{pmatrix} \theta\,(x = 0, p) \\ \phi\,(x = 0, p) \end{pmatrix} = \begin{pmatrix} \theta_E \\ \phi_E \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \theta_S \\ \phi_S \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \theta\,(x = e, p) \\ \phi\,(x = e, p) \end{pmatrix}$$

for several boundary conditions (Dirac heat flux, insulated back side, imposed or Heaviside temperature, etc.). For instance, the heat flux for a multilayer medium with the Heaviside temperature at the front side and the imposed temperature at the back side is given by the following formulas:

$$\phi_S = \frac{1}{bp} = \frac{1}{\displaystyle Rp \sum_{k \text{ odd}}^{n} C_n^k (RCp)^{(k-1)/2}}, \quad \phi_E = \frac{d}{bp} = \frac{\displaystyle\sum_{k \text{ even}}^{n} C_n^k (RCp)^{k/2}}{\displaystyle Rp \sum_{k \text{ odd}}^{n} C_n^k (RCp)^{(k-1)/2}} = d\phi_S\,.$$

TABLE 6. Expressions for Elements of the Transfer Matrix in the Transient Case ($\xi = RCp$)

Polynomial	n	
	10	20
P_1^n	$(1+\xi^5) + 45(\xi+\xi^4) + 210(\xi^2+\xi^3)$	$(1+\xi^{10}) + 190(\xi+\xi^9) + 4845(\xi^2+\xi^8) +$ $+ 38760(\xi^3+\xi^7) + 125\,970(\xi^4+\xi^6) + 184\,756\xi^5$
P_2^n	$10(1+\xi^4) + 120(\xi+\xi^3) + 252\xi^2$	$20(1+\xi^9) + 1140(\xi+\xi^8) + 15\;504(\xi^2+\xi^7) +$ $+77\,520(\xi^3+\xi^6) + 167\,960(\xi^4+\xi^5)$

TABLE 7. Heat Flux and Temperature as Functions of the Transfer-Matrix Elements for Different Boundary Conditions

Boundary conditions		ϕ_E	θ_E	ϕ_S	θ_S
Front side	Back side				
Dirac heat flux	Insulated	1	$\frac{a}{c}$	0	$\frac{1}{c}$
Heaviside heat flux	Insulated	$\frac{1}{p}$	$\frac{a}{cp}$	0	$\frac{1}{pc}$
Heaviside heat flux	Imposed temperature	$\frac{1}{p}$	$\frac{b}{dp}$	$\frac{1}{dp}$	0
Heaviside temperature	Imposed temperature	$\frac{d}{bp}$	$\frac{1}{p}$	$\frac{1}{bp}$	0
Heaviside temperature	Insulated	$\frac{c}{ap}$	$\frac{1}{p}$	0	$\frac{1}{ap}$

Then a numerical algorithm permits one to obtain the heat flux as a function of time.

Let us consider the second example. The temperature of the insulated back side of a multilayer medium with the Dirac heat flux at the front side is given by the formula

$$\theta_S = \frac{1}{c}\,\frac{1}{Cp \displaystyle\sum_{k\ \mathrm{odd}}^{n} C_n^k\,(RCp)^{(k-1)/2}}\,.$$

This gives for a multilayer consisting of eleven steel layers (see Table 6)

$$\theta_S = \frac{1}{c} = \frac{1}{Cp\,(10\,(1 + R^4C^4p^4) + 120\,(RCp + R^3C^3p^3) + 252R^2C^2p^2)} \underset{p\to 0}{\approx} \frac{1}{10Cp\,(1 + 12RCp)}\,.$$

In this case, it is possible to obtain an analytical approximate expression for the transient temperature (it could be useful, for instance, for parameter estimation or in optimization):

$$\theta_S \approx \frac{1}{10Cp} - \frac{12R}{10}\,\frac{1}{1 + 12RCp} \to \theta_S\,(t) \approx \frac{1}{10C}\,(H\,(t) - \exp\,(-\,t/12RC))\,.$$

Conclusions. Heat transfer in the case of a multilayer medium under several boundary conditions is investigated. The study is not limited to the case of two layers: an explicit matrix is given in the general case. Then the modeling is used to treat the case of a stratified steel mould (with ten or twenty brazed layers). Not only steady-state but also transient results are obtained. The absolute differences between the equivalent thermal resistance of the multilayer and the thermal resistance of the medium without brazing are very small (for heat transfer orthogonal to the layers, they are smaller than for parallel ones). The general formula for the matrix is then established in the case of brazed layers to obtain the transient behavior of the multilayer.

NOTATION

A, B, M, N, Y, matrices; a, b, c, d, elements of matrices; c_A and c_B, specific heat capacities; e, thickness; H, Heaviside function; n, number of brazed layers; p, Laplace variable; R, equivalent resistance; S, surface area; T, temperature; t, time; $\mathbf{V}_i$, eigenvector; x, Cartesian coordinate; α, thermal diffusivity; θ, temperature image; λ, thermal conductivity; λ_i, eigenvalue; ρ_A and ρ_B, densities of layers A and B; ϕ, heat flux in the image domain. Subscripts: A and B, layers A and B; bi, bilayer; E, at $x = 0$; S, at $x = e_A + e_B$; s, steel.

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